1 Introduction

We always denote by $X$ our universe, i.e. all the sets we shall consider are subsets of $X$.

Recall some standard notation. $2^X$ everywhere denotes the set of all subsets of a given set $X$. If $A \cap B = \emptyset$ then we often write $A \cup B$ rather than $A \cup B$, to underline the disjointness. The complement (in $X$) of a set $A$ is denoted by $A^c$. By $A \triangle B$ the symmetric difference of $A$ and $B$ is denoted, i.e. $A \triangle B = (A \setminus B) \cup (B \setminus A)$. Letters $i, j, k$ always denote positive integers. The sign $↾$ is used for restriction of a function (operator etc.) to a subset (subspace).

1.1 The Riemann integral

Recall how to construct the Riemannian integral. Let $f : [a, b] \rightarrow \mathbb{R}$. Consider a partition $\pi$ of $[a, b]$: 

$$ a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b $$

and set $\Delta x_k = x_{k+1} - x_k$, $|\pi| = \max\{\Delta x_k : k = 0, 1, \ldots, n - 1\}$, $m_k = \inf\{f(x) : x \in [x_k, x_{k+1}]\}$, $M_k = \sup\{f(x) : x \in [x_k, x_{k+1}]\}$. Define the upper and lower Riemann—Darboux sums

$$ \underline{s}(f, \pi) = \sum_{k=0}^{n-1} m_k \Delta x_k, \quad \overline{s}(f, \pi) = \sum_{k=0}^{n-1} M_k \Delta x_k. $$

One can show (the Darboux theorem) that the following limits exist

$$ \lim_{|\pi| \rightarrow 0} \underline{s}(f, \pi) = \sup_{\pi} \underline{s}(f, \pi) = \int_a^b f \, dx $$

$$ \lim_{|\pi| \rightarrow 0} \overline{s}(f, \pi) = \inf_{\pi} \overline{s}(f, \pi) = \int_a^b f \, dx. $$
Clearly, 
\[ s(f, \pi) \leq \int_a^b f \, dx \leq \int_a^b \bar{s}(f) \leq \bar{s}(f, \pi) \]
for any partition \( \pi \).

The function \( f \) is said to be Riemann integrable on \([a, b]\) if the upper and lower integrals are equal. The common value is called Riemann integral of \( f \) on \([a, b]\).

The functions cannot have a large set of points of discontinuity. More precisely this will be stated further.

### 1.2 The Lebesgue integral

It allows to integrate functions from a much more general class. First, consider a very useful example. For \( f, g \in C[a, b] \), two continuous functions on the segment \([a, b] = \{ x \in \mathbb{R} : a \leq x \leq b \}\) put
\[
\rho_1(f, g) = \max_{a \leq x \leq b} |f(x) - g(x)|,
\]
\[
\rho_2(f, g) = \int_a^b |f(x) - g(x)| \, dx.
\]

Then \((C[a, b], \rho_1)\) is a complete metric space, when \((C[a, b], \rho_2)\) is not. To prove the latter statement, consider a family of functions \( \{\varphi_n\}_{n=1}^{\infty} \) as drawn on Fig.1. This is a Cauchy sequence with respect to \( \rho_2 \). However, the limit does not belong to \( C[a, b] \).
2 Systems of Sets

Definition 2.1 A ring of sets is a non-empty subset in $2^X$ which is closed with respect to the operations $\cup$ and $\setminus$.

Proposition. Let $\mathcal{R}$ be a ring of sets. Then $\emptyset \in \mathcal{R}$.

Proof. Since $\mathcal{R} \neq \emptyset$, there exists $A \in \mathcal{R}$. Since $\mathcal{R}$ contains the difference of every two its elements, one has $A \setminus A = \emptyset \in \mathcal{R}$. ■

Examples.

1. The two extreme cases are $\mathcal{R} = \{\emptyset\}$ and $\mathcal{R} = 2^X$.

2. Let $X = \mathbb{R}$ and denote by $\mathcal{R}$ all finite unions of semi-segments $[a, b)$.

Definition 2.2 A semi-ring is a collection of sets $\mathcal{P} \subset 2^X$ with the following properties:

1. If $A, B \in \mathcal{P}$ then $A \cap B \in \mathcal{P}$;
2. For every $A, B \in \mathcal{P}$ there exists a finite disjoint collection $(C_j)\ j = 1, 2, \ldots, n$ of sets (i.e. $C_i \cap C_j = \emptyset$ if $i \neq j$) such that

$$A \setminus B = \bigsqcup_{j=1}^{n} C_j.$$ 

Example. Let $X = \mathbb{R}$, then the set of all semi-segments, $[a, b)$, forms a semi-ring.

Definition 2.3 An algebra (of sets) is a ring of sets containing $X \in 2^X$.

Examples.

1. $\{\emptyset, X\}$ and $2^X$ are the two extreme cases (note that they are different from the corresponding cases for rings of sets).

2. Let $X = [a, b)$ be a fixed interval on $\mathbb{R}$. Then the system of finite unions of subintervals $[\alpha, \beta) \subset [a, b)$ forms an algebra.

3. The system of all bounded subsets of the real axis is a ring (not an algebra).

Remark. $\mathfrak{A}$ is algebra if (i) $A, B \in \mathfrak{A} \implies A \cup B \in \mathfrak{A}$, (ii) $A \in \mathfrak{A} \implies A^c \in \mathfrak{A}$.

Indeed, 1) $A \cap B = (A^c \cup B^c)^c$; 2) $A \setminus B = A \cap B^c$.

Definition 2.4 A $\sigma$-ring (a $\sigma$-algebra) is a ring (an algebra) of sets which is closed with respect to all countable unions.

Definition 2.5 A ring (an algebra, a $\sigma$-algebra) of sets, $\mathfrak{A}(\mathfrak{U})$ generated by a collection of sets $\mathfrak{U} \subset 2^X$ is the minimal ring (algebra, $\sigma$-algebra) of sets containing $\mathfrak{U}$.

In other words, it is the intersection of all rings (algebras, $\sigma$-algebras) of sets containing $\mathfrak{U}$.
3 Measures

Let $X$ be a set, $\mathfrak{A}$ an algebra on $X$.

**Definition 3.1** A function $\mu : \mathfrak{A} \rightarrow \mathbb{R}_+ \cup \{\infty\}$ is called a measure if

1. $\mu(A) \geq 0$ for any $A \in \mathfrak{A}$ and $\mu(\emptyset) = 0$;
2. if $(A_i)_{i \geq 1}$ is a disjoint family of sets in $\mathfrak{A}$ ( $A_i \cap A_j = \emptyset$ for any $i \neq j$) such that $\bigcup_{i=1}^{\infty} A_i \in \mathfrak{A}$, then

   $$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

The latter important property, is called *countable additivity* or *σ-additivity* of the measure $\mu$.

Let us state now some elementary properties of a measure. Below till the end of this section $\mathfrak{A}$ is an algebra of sets and $\mu$ is a measure on it.

1. (Monotonicity of $\mu$) If $A, B \in \mathfrak{A}$ and $B \subset A$ then $\mu(B) \leq \mu(A)$.
   
   *Proof.* $A = (A \setminus B) \cup B$ implies that
   $$\mu(A) = \mu(A \setminus B) + \mu(B).$$

   Since $\mu(A \setminus B) \geq 0$ it follows that $\mu(A) \geq \mu(B)$.

2. (Subtractivity of $\mu$). If $A, B \in \mathfrak{A}$ and $B \subset A$ and $\mu(B) < \infty$ then $\mu(A \setminus B) = \mu(A) - \mu(B)$.
   
   *Proof.* In 1) we proved that
   $$\mu(A) = \mu(A \setminus B) + \mu(B).$$

   If $\mu(B) < \infty$ then
   $$\mu(A) - \mu(B) = \mu(A \setminus B).$$

3. If $A, B \in \mathfrak{A}$ and $\mu(A \cap B) < \infty$ then $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$.
   
   *Proof.* $A \cap B \subset A, A \cap B \subset B$, therefore
   $$A \cup B = (A \setminus (A \cap B)) \cup B.$$

   Since $\mu(A \cap B) < \infty$, one has
   $$\mu(A \cup B) = (\mu(A) - \mu(A \cap B)) + \mu(B).$$
4. (Semi-additivity of $\mu$). If $(A_i)_{i \geq 1} \subset \mathcal{A}$ such that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$ then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

Proof. First let us prove that

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mu(A_i).$$

Note that the family of sets

$$B_1 = A_1$$
$$B_2 = A_2 \setminus A_1$$
$$B_3 = A_3 \setminus (A_1 \cup A_2)$$
$$\ldots$$
$$B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i$$

is disjoint and $\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i$. Moreover, since $B_i \subset A_i$, we see that $\mu(B_i) \leq \mu(A_i)$. Then

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \mu\left(\bigcup_{i=1}^{n} B_i\right) = \sum_{i=1}^{n} \mu(B_i) \leq \sum_{i=1}^{n} \mu(A_i).$$

Now we can repeat the argument for the infinite family using $\sigma$-additivity of the measure.

### 3.1 Continuity of a measure

**Theorem 3.1** Let $\mathcal{A}$ be an algebra, $(A_i)_{i \geq 1} \subset \mathcal{A}$ a monotonically increasing sequence of sets $(A_i \subset A_{i+1})$ such that $\bigcup_{i\geq 1} \in \mathcal{A}$. Then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n \to \infty} \mu(A_n).$$

**Proof.** 1). If for some $n_0$ $\mu(A_{n_0}) = +\infty$ then $\mu(A_n) = +\infty \ \forall n \geq n_0$ and $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = +\infty$.

2). Let now $\mu(A_i) < \infty \ \forall i \geq 1$. 


Then
\[
\mu(\bigcup_{i=1}^{\infty} A_i) = \mu(A_1 \cup (A_2 \setminus A_1) \cup \ldots \cup (A_n \setminus A_{n-1}) \cup \ldots)
\]
\[= \mu(A_1) + \sum_{k=2}^{\infty} \mu(A_k \setminus A_{k-1})
\]
\[= \mu(A_1) + \lim_{n \to \infty} \sum_{k=2}^{n} (\mu(A_k) - \mu(A_{k-1})) = \lim_{n \to \infty} \mu(A_n).
\]

### 3.2 Outer measure

Let \( \mathcal{A} \) be an algebra of subsets of \( X \) and \( \mu \) a measure on it. Our purpose now is to extend \( \mu \) to as many elements of \( 2^X \) as possible.

An arbitrary set \( A \subset X \) can be always covered by sets from \( \mathcal{A} \), i.e. one can always find \( E_1, E_2, \ldots \in \mathcal{A} \) such that \( \bigcup_{i=1}^{\infty} E_i \supset A \). For instance, \( E_1 = X, E_2 = E_3 = \ldots = \emptyset \).

**Definition 3.2** For \( A \subset X \) its outer measure is defined by
\[
\mu^*(A) = \inf \sum_{i=1}^{\infty} \mu(E_i)
\]
where the infimum is taken over all \( \mathcal{A} \)-coverings of the set \( A \), i.e. all collections \( (E_i), \ E_i \in \mathcal{A} \) with \( \bigcup_i E_i \supset A \).

**Remark.** The outer measure always exists since \( \mu(A) \geq 0 \) for every \( A \in \mathcal{A} \).

**Example.** Let \( X = \mathbb{R}^2 \), \( \mathcal{A} = \mathcal{B}(\mathbb{R}) \), -\( \sigma \)-algebra generated by \( \mathbb{P} \), \( \mathbb{P} = \{[a, b) \times \mathbb{R}^1\} \). Thus \( \mathcal{A} \) consists of countable unions of strips like one drawn on the picture. Put \( \mu([a, b) \times \mathbb{R}^1) = b - a \). Then, clearly, the outer measure of the unit disc \( x^2 + y^2 \leq 1 \) is equal to 2. The same value is for the square \( |x| \leq 1, \ |y| \leq 1 \).

**Theorem 3.2** For \( A \in \mathcal{A} \) one has \( \mu^*(A) = \mu(A) \).

In other words, \( \mu^* \) is an extension of \( \mu \).

**Proof.** 1. \( A \) is its own covering. This implies \( \mu^*(A) \leq \mu(A) \).

2. By definition of infimum, for any \( \varepsilon > 0 \) there exists a \( \mathcal{A} \)-covering \( (E_i) \) of \( A \) such that \( \sum_i \mu(E_i) < \mu^*(A) + \varepsilon \). Note that
\[
A = A \cap (\bigcup_i E_i) = \bigcup_i (A \cap E_i).
\]
Using consequently $\sigma$-semiadditivity and monotonicity of $\mu$, one obtains:

$$\mu(A) \leq \sum_i \mu(A \cap E_i) \leq \sum_i \mu(E_i) < \mu^*(A) + \varepsilon.$$ 

Since $\varepsilon$ is arbitrary, we conclude that $\mu(A) \leq \mu^*(A)$. $lacksquare$

It is evident that $\mu^*(A) \geq 0$, $\mu^*(\emptyset) = 0$ (Check!).

**Lemma.** Let $\mathcal{A}$ be an algebra of sets (not necessary $\sigma$-algebra), $\mu$ a measure on $\mathcal{A}$. If there exists a set $A \in \mathcal{A}$ such that $\mu(A) < \infty$, then $\mu(\emptyset) = 0$.

**Proof.** $\mu(A \setminus A) = \mu(A) - \mu(A) = 0$. $lacksquare$

Therefore the property $\mu(\emptyset) = 0$ can be substituted with the existence in $\mathcal{A}$ of a set with a finite measure.

**Theorem 3.3** (*Monotonicity of outer measure*). If $A \subset B$ then $\mu^*(A) \leq \mu^*(B)$.

**Proof.** Any covering of $B$ is a covering of $A$. $lacksquare$

**Theorem 3.4** (*$\sigma$-semiadditivity of $\mu^*$*). $\mu^*(\bigcup_{j=1}^{\infty} A_j) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$. 

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Proof. If the series in the right-hand side diverges, there is nothing to prove. So assume that it is convergent.

By the definition of outer measur for any $\varepsilon > 0$ and for any $j$ there exists an $A$-covering $\bigcup E_{kj} \supset A_j$ such that

$$\sum_{k=1}^{\infty} \mu(E_{kj}) < \mu^*(A_j) + \frac{\varepsilon}{2^j}.$$  

Since

$$\bigcup_{j,k=1}^{\infty} E_{kj} \supset \bigcup_{j=1}^{\infty} A_j,$$

the definition of $\mu^*$ implies

$$\mu^*(\bigcup_{j=1}^{\infty} A_j) \leq \sum_{j,k=1}^{\infty} \mu(E_{kj})$$

and therefore

$$\mu^*(\bigcup_{j=1}^{\infty} A_j) < \sum_{j=1}^{\infty} \mu^*(A_j) + \varepsilon.$$

$$\blacksquare$$

3.3 Measurable Sets

Let $\mathcal{A}$ be an algebra of subsets of $X$, $\mu$ a measure on it, $\mu^*$ the outer measure defined in the previous section.

Definition 3.3 A $\subset X$ is called a measurable set (by Carathéodory) if for any $E \subset X$ the following relation holds:

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Denote by $\tilde{\mathcal{A}}$ the collection of all set which are measurable by Carathéodory and set $\tilde{\mu} = \mu^* \upharpoonright \tilde{\mathcal{A}}$.

Remark Since $E = (E \cap A) \cup (E \cap A^c)$, due to semiadditivity of the outer measure

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Theorem 3.5 $\tilde{\mathcal{A}}$ is a $\sigma$-algebra containing $\mathcal{A}$, and $\tilde{\mu}$ is a measure on $\tilde{\mathcal{A}}$. 9
Proof. We divide the proof into several steps.

1. **If** $A, B \in \tilde{\mathfrak{A}}$ **then** $A \cup B \in \tilde{\mathfrak{A}}$.

   By the definition one has
   \[
   \mu^*(E) = \mu^*(E \cap B) + \mu^*(E \cap B^c). \tag{1}
   \]

   Take $E \cap A$ instead of $E$:
   \[
   \mu^*(E \cap A) = \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c). \tag{2}
   \]

   Then put $E \cap A^c$ in (1) instead of $E$:
   \[
   \mu^*(E \cap A^c) = \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c). \tag{3}
   \]

   Add (2) and (3):
   \[
   \mu^*(E) = \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c). \tag{4}
   \]

   Substitute $E \cap (A \cup B)$ in (4) instead of $E$. Note that
   \[
   1) \quad E \cap (A \cup B) \cap A \cap B = E \cap A \cap B \\
   2) \quad E \cap (A \cup B) \cap A^c \cap B = E \cap A^c \cap B \\
   3) \quad E \cap (A \cup B) \cap A \cap B^c = E \cap A \cap B^c \\
   4) \quad E \cap (A \cup B) \cap A^c \cap B^c = \emptyset.
   \]

   One has
   \[
   \mu^*(E \cap (A \cup B)) = \mu^*(E \cap A \cap B) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A \cap B^c). \tag{5}
   \]

   From (4) and (5) we have
   \[
   \mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c).
   \]

2. **If** $A \in \tilde{\mathfrak{A}}$ **then** $A^c \in \tilde{\mathfrak{A}}$.

   The definition of measurable set is symmetric with respect to $A$ and $A^c$.

   Therefore $\tilde{\mathfrak{A}}$ is an algebra of sets.

3. Let $A, B \in \mathfrak{A}$, $A \cap B = \emptyset$. From (5)
   \[
   \mu^*(E \cap (A \cup B)) = \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A \cap B^c) = \mu^*(E \cap B) + \mu^*(E \cap A).
   \]
4. $\tilde{\mathcal{A}}$ is a $\sigma$-algebra.

From the previous step, by induction, for any finite disjoint collection $(B_j)$ of sets:

$$\mu^*(E \cap (\bigcup_{j=1}^{n} B_j)) = \sum_{j=1}^{n} \mu^*(E \cap B_j). \quad (6)$$

Let $A = \bigcup_{j=1}^{\infty} A_j, A_j \in \mathcal{A}$. Then $A = \bigcup_{j=1}^{\infty} B_j, B_j = A_j \setminus \bigcup_{k=1}^{j-1} A_k$ and

$B_i \cap B_j = \emptyset$ ($i \neq j$). It suffices to prove that

$$\mu^*(E) \geq \mu^*(E \cap (\bigcup_{j=1}^{\infty} B_j)) + \mu^*(E \cap (\bigcup_{j=1}^{\infty} B_j)^c). \quad (7)$$

Indeed, we have already proved that $\mu^*$ is $\sigma$-semi-additive.

Since $\tilde{\mathcal{A}}$ is an algebra, it follows that $\bigcup_{j=1}^{n} B_j \in \tilde{\mathcal{A}} (\forall n \in \mathbb{N})$ and the following inequality holds for every $n$:

$$\mu^*(E) \geq \mu^*(E \cap (\bigcup_{j=1}^{n} B_j)) + \mu^*(E \cap (\bigcup_{j=1}^{n} B_j)^c). \quad (8)$$

Since $E \cap (\bigcup_{j=1}^{\infty} B_j)^c \subset E \cap (\bigcup_{j=1}^{n} B_j)^c$, by monotonicity of the measure and (8),

$$\mu^*(E) \geq \sum_{j=1}^{n} \mu^*(E \cap B_j) + \mu^*(E \cap A^c). \quad (9)$$

Passing to the limit we get

$$\mu^*(E) \geq \sum_{j=1}^{\infty} \mu^*(E \cap B_j) + \mu^*(E \cap A^c). \quad (10)$$

Due to semiaadditivity

$$\mu^*(E \cap A) = \mu^*(E \cap (\bigcup_{j=1}^{\infty} B_j)) = \mu^*(\bigcup_{j=1}^{\infty}(E \cap B_j)) \leq \sum_{j=1}^{\infty} \mu^*(E \cap B_j).$$

Compare this with (10):

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Thus, $A \in \tilde{\mathcal{A}}$, which means that $\tilde{\mathcal{A}}$ is a $\sigma$-algebra.

5. $\hat{\mu} = \mu^* | \tilde{\mathcal{A}}$ is a measure.
We need to prove only $\sigma$-additivity. Let $E = \bigcup_{j=1}^{\infty} A_j$. From (10) we get

$$\mu^*(\bigcup_{j=1}^{\infty} A_j) \geq \sum_{j=1}^{\infty} \mu^*(A_j).$$

The opposite inequality follows from $\sigma$-semiadditivity of $\mu^*$.

6. $\mathfrak{A} \supseteq \mathfrak{A}$.

Let $A \in \mathfrak{A}$, $E \subset X$. We need to prove:

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c). \quad (11)$$

If $E \in \mathfrak{A}$ then (11) is clear since $E \cap A$ and $E \cap A^c$ are disjoint and both belong to $\mathfrak{A}$ where $\mu^* = \mu$ and so is additive.

For $E \subset X$ for $\forall \varepsilon > 0$ there exists a $\mathfrak{A}$-covering $(E_j)$ of $E$ such that

$$\mu^*(E) + \varepsilon > \sum_{j=1}^{\infty} \mu(E_j). \quad (12)$$

Now, since $E_j = (E_j \cap A) \cup (E_j \cap A^c)$, one has

$$\mu(E_j) = \mu(E_j \cap A) + \mu(E_j \cap A^c)$$

and also

$$E \cap A \subset \bigcup_{j=1}^{\infty} (E_j \cap A)$$

$$E \cap A^c \subset \bigcup_{j=1}^{\infty} (E_j \cap A^c)$$

By monotonicity and $\sigma$-semiadditivity

$$\mu^*(E \cap A) \leq \sum_{j=1}^{\infty} \mu(E_j \cap A),$$

$$\mu^*(E \cap A^c) \leq \sum_{j=1}^{\infty} \mu(E_j \cap A^c).$$

Adding the last two inequalities we obtain

$$\mu^*(E \cap A) + \mu^*(E \cap A^c) \leq \sum_{j=1}^{\infty} \mu^*(E_j) < \mu^*(E) + \varepsilon.$$ 

Since $\varepsilon > 0$ is arbitrary, (11) is proved.

The following theorem is a direct consequence of the previous one.
Theorem 3.6 Let $\mathfrak{A}$ be an algebra of subsets of $X$ and $\mu$ be a measure on it. Then there exists a $\sigma$-algebra $\mathfrak{A}_1 \supset \mathfrak{A}$ and a measure $\mu_1$ on $\mathfrak{A}_1$ such that $\mu_1 \upharpoonright \mathfrak{A} = \mu$.

**Remark.** Consider again an algebra $\mathfrak{A}$ of subsets of $X$. Denote by $\mathfrak{A}_\sigma$ the generated $\sigma$-algebra and construct the extension $\mu_\sigma$ of $\mu$ on $\mathfrak{A}_\sigma$. This extension is called minimal extension of measure.

Since $\tilde{\mathfrak{A}} \supset \mathfrak{A}$ therefore $\mathfrak{A}_\sigma \subset \tilde{\mathfrak{A}}$. Hence one can set $\mu_\sigma = \tilde{\mu} \upharpoonright \mathfrak{A}_\sigma$. Obviously $\mu_\sigma$ is a minimal extension of $\mu$. It always exists. On can also show (see below) that this extension is unique.

Theorem 3.7 Let $\mu$ be a measure on an algebra $\mathfrak{A}$ of subsets of $X$, $\mu^*$ the corresponding outer measure. If $\mu^*(A) = 0$ for a set $A \subset X$ then $A \in \tilde{\mathfrak{A}}$ and $\tilde{\mu}(A) = 0$.

**Proof.** Clearly, it suffices to prove that $A \in \tilde{\mathfrak{A}}$. Further, it suffices to prove that $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$. The latter statement follows from monotonicity of $\mu^*$. Indeed, one has $\mu^*(E \cap A) \leq \mu^*(A) = 0$ and $\mu^*(E \cap A^c) \leq \mu^*(E)$. ■

Definition 3.4 A measure $\mu$ on an algebra of sets $\mathfrak{A}$ is called complete if conditions $B \subset A, A \in \mathfrak{A}, \mu(A) = 0$ imply $B \in \mathfrak{A}$ and $\mu(B) = 0$.

Corollary. $\tilde{\mu}$ is a complete measure.

Definition 3.5 A measure $\mu$ on an algebra $\mathfrak{A}$ is called finite if $\mu(X) < \infty$. It is called $\sigma$-finite if there is an increasing sequence $(F_j)_{j \geq 1} \subset \mathfrak{A}$ such that $X = \bigcup_j F_j$ and $\mu(F_j) < \infty$ $\forall j$.

Theorem 3.8 Let $\mu$ be a $\sigma$-finite measure on an algebra $\mathfrak{A}$. Then there exist a unique extension of $\mu$ to a measure on $\tilde{\mathfrak{A}}$.

**Proof.** It suffices to show uniqueness. Let $\nu$ be another extension of $\mu$ ($\nu \upharpoonright \mathfrak{A} = \mu \upharpoonright \mathfrak{A}$).

First, let $\mu$ (and therefore $\nu, \mu^*$) be finite. Let $A \in \tilde{\mathfrak{A}}$. Let $(E_j) \subset \mathfrak{A}$ such that $A \subset \bigcup_j E_j$. We have

$$\nu(A) \leq \nu(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \nu(E_j) = \sum_{j=1}^{\infty} \mu(E_j).$$

Therefore

$$\nu(A) \leq \mu^*(A) \quad \forall A \in \tilde{\mathfrak{A}}.$$
Since $\mu^*$ and $\nu$ are additive (on $\tilde{\mathfrak{A}}$) it follows that

$$\mu^*(A) + \mu^*(A^c) = \nu(A) + \nu(A^c).$$

The terms in the RHS are finite and $\nu(A) \leq \mu^*(A)$, $\nu(A^c) \leq \mu^*(A^c)$. From this we infer that

$$\nu(A) = \mu^*(A) \ \forall A \in \tilde{\mathfrak{A}}.$$

Now let $\mu$ be $\sigma$-finite, $(F_j)$ be an increasing sequence of sets from $\mathfrak{A}$ such that $\mu(F_j) < \infty \ \forall j$ and $X = \bigcup_{j=1}^{\infty} F_j$. From what we have already proved it follows that

$$\mu^*(A \cap F_j) = \nu(A \cap F_j) \ \forall A \in \tilde{\mathfrak{A}}.$$

Therefore

$$\mu^*(A) = \lim_{j \to \infty} \mu^*(A \cap F_j) = \lim_{j \to \infty} \nu(A \cap F_j) = \nu(A). \ \Box$$

**Theorem 3.9 (Continuity of measure).** Let $\mathfrak{A}$ be a $\sigma$-algebra with a measure $\mu$, $\{A_j\} \subset \mathfrak{A}$ a monotonically increasing sequence of sets. Then

$$\mu(\bigcup_{j=1}^{\infty} A_j) = \lim_{j \to \infty} \mu(A_j).$$

**Proof.** One has:

$$A = \bigcup_{j=1}^{\infty} A_j = \bigcup_{j=2}^{\infty} (A_{j+1} \setminus A_j) \cup A_1.$$

Using $\sigma$-additivity and subtractivity of $\mu$,

$$\mu(A) = \sum_{j=1}^{\infty} (\mu(A_{j+1}) - \mu(A_j)) + \mu(A_1) = \lim_{j \to \infty} \mu(A_j). \ \Box$$

Similar assertions for a decreasing sequence of sets in $\mathfrak{A}$ can be proved using de Morgan formulas.

**Theorem 3.10** Let $A \in \tilde{\mathfrak{A}}$. Then for any $\varepsilon > 0$ there exists $A_\varepsilon \in \mathfrak{A}$ such that $\mu^*(A \Delta A_\varepsilon) < \varepsilon$.

**Proof.** 1. For any $\varepsilon > 0$ there exists an $\mathfrak{A}$ cover $\bigcup E_j \supset A$ such that

$$\sum_j \mu(E_j) < \mu^*(A) + \frac{\varepsilon}{2} = \tilde{\mu}(A) + \frac{\varepsilon}{2}.$$
On the other hand,

$$\sum_{j} \mu(E_j) \geq \tilde{\mu}(\bigcup_{j} E_j).$$

The monotonicity of $\tilde{\mu}$ implies

$$\tilde{\mu}(\bigcup_{j=1}^{\infty} E_j) = \lim_{n \to \infty} \tilde{\mu}(\bigcup_{j=1}^{n} E_j),$$

hence there exists a positive integer $N$ such that

$$\tilde{\mu}(\bigcup_{j=1}^{\infty} E_j) - \tilde{\mu}(\bigcup_{j=1}^{N} E_j) < \frac{\varepsilon}{2}.$$  \hspace{1cm} (13)

2. Now, put

$$A_{\varepsilon} = \bigcup_{j=1}^{N} E_j$$

and prove that $\mu^*(A \Delta A_{\varepsilon}) < \varepsilon$.

2a. Since

$$A \subset \bigcup_{j=1}^{\infty} E_j,$$

one has

$$A \setminus A_{\varepsilon} \subset \bigcup_{j=1}^{\infty} E_j \setminus A_{\varepsilon}.$$ 

Since

$$A_{\varepsilon} \subset \bigcup_{j=1}^{\infty} E_j,$$

one can use the monotonicity and subtractivity of $\tilde{\mu}$. Together with estimate (13), this gives

$$\tilde{\mu}(A \setminus A_{\varepsilon}) \leq \tilde{\mu}(\bigcup_{j=1}^{\infty} E_j \setminus A_{\varepsilon}) < \frac{\varepsilon}{2}.$$ 

2b. The inclusion

$$A_{\varepsilon} \setminus A \subset \bigcup_{j=1}^{\infty} E_j \setminus A$$

implies

$$\tilde{\mu}(A_{\varepsilon} \setminus A) \leq \tilde{\mu}(\bigcup_{j=1}^{\infty} E_j \setminus A) = \tilde{\mu}(\bigcup_{j=1}^{\infty} E_j) - \tilde{\mu}(A) < \frac{\varepsilon}{2}.$$
Here we used the same properties of $\tilde{\mu}$ as above and the choice of the cover $(E_j)$.

3. Finally,

$$\tilde{\mu}(A \triangle A_\varepsilon) \leq \tilde{\mu}(A \setminus A_\varepsilon) + \tilde{\mu}(A_\varepsilon \setminus A).$$

■
4 Monotone Classes
and Uniqueness of Extension of Measure

Definition 4.1 A collection of sets, \( \mathcal{M} \) is called a monotone class if together with any monotone sequence of sets \( \mathcal{M} \) contains the limit of this sequence.

Example. Any \( \sigma \)-ring. (This follows from the Exercise 1. below).

Exercises.

1. Prove that any \( \sigma \)-ring is a monotone class.
2. If a ring is a monotone class, then it is a \( \sigma \)-ring.

We shall denote by \( \mathcal{M}(\mathcal{K}) \) the minimal monotone class containing \( \mathcal{K} \).

Theorem 4.1 Let \( \mathcal{K} \) be a ring of sets, \( \mathcal{K}_\sigma \) the \( \sigma \)-ring generated by \( \mathcal{K} \). Then \( \mathcal{M}(\mathcal{K}) = \mathcal{K}_\sigma \).

Proof. 1. Clearly, \( \mathcal{M}(\mathcal{K}) \subset \mathcal{K}_\sigma \). Now, it suffices to prove that \( \mathcal{M}(\mathcal{K}) \) is a ring. This follows from the Exercise (2) above and from the minimality of \( \mathcal{K}_\sigma \).

2. \( \mathcal{M}(\mathcal{K}) \) is a ring.

2a. For \( B \subset X \), set

\[
\mathcal{K}_B = \{ A \subset X : A \cup B, A \cap B, A \setminus B, B \setminus A \in \mathcal{M}(\mathcal{K}) \}.
\]

This definition is symmetric with respect to \( A \) and \( B \), therefore \( A \in \mathcal{K}_B \) implies \( B \in \mathcal{K}_A \).

2b. \( \mathcal{K}_B \) is a monotone class.

Let \( (A_j) \subset \mathcal{K}_B \) be a monotonically increasing sequence. Prove that the union, \( A = \bigcup A_j \) belongs to \( \mathcal{K}_B \).

Since \( A_j \in \mathcal{K}_B \), one has \( A_j \cup B \in \mathcal{K}_B \), and so

\[
A \cup B = \bigcup_{j=1}^{\infty} (A_j \cup B) \in \mathcal{M}(\mathcal{K}).
\]

In the same way,

\[
A \setminus B = (\bigcup_{j=1}^{\infty} A_j) \setminus B = \bigcup_{j=1}^{\infty} (A_j \setminus B) \in \mathcal{M}(\mathcal{K});
\]
\[ B \setminus A = B \setminus \left( \bigcup_{j=1}^{\infty} A_j \right) = \bigcap_{j=1}^{\infty} (B \setminus A_j) \in \mathcal{M}(\mathcal{K}). \]

Similar proof is for the case of decreasing sequence \((A_j)\).

2c. If \(B \in \mathcal{K}\) then \(\mathcal{M}(\mathcal{K}) \subset \mathcal{K}_B\).

Obviously, \(\mathcal{K} \subset \mathcal{K}_B\). Together with minimality of \(\mathcal{M}(\mathcal{K})\), this implies \(\mathcal{M}(\mathcal{K}) \subset \mathcal{K}_B\).

2d. If \(B \in \mathcal{M}(\mathcal{K})\) then \(\mathcal{M}(\mathcal{K}) \subset \mathcal{K}_B\).

Let \(A \in \mathcal{K}\). Then \(\mathcal{M}(\mathcal{K}) \subset \mathcal{K}_A\). Thus if \(B \in \mathcal{M}(\mathcal{K})\), one has \(B \in \mathcal{K}_A\), so \(A \in \mathcal{K}_B\).

Hence what we have proved is \(\mathcal{K} \subset \mathcal{K}_B\). This implies \(\mathcal{M}(\mathcal{K}) \subset \mathcal{K}_B\).

2e. It follows from 2a. — 2d. that if \(A, B \in \mathcal{M}(\mathcal{K})\) then \(A \in \mathcal{K}_B\) and so \(A \cup B, A \cap B, A \setminus B\) and \(B \setminus A\) all belong to \(\mathcal{M}(\mathcal{K})\). ■

**Theorem 4.2** Let \(\mathcal{A}\) be an algebra of sets, \(\mu\) and \(\nu\) two measures defined on the \(\sigma\)-algebra \(\mathcal{A}_\sigma\) generated by \(\mathcal{A}\). Then \(\mu \upharpoonright \mathcal{A} = \nu \upharpoonright \mathcal{A}\) implies \(\mu = \nu\).

**Proof.** Choose \(A \in \mathcal{A}_\sigma\), then \(A = \lim_{n \to \infty} A_n\), \(A_n \in \mathcal{A}\), for \(\mathcal{A}_\sigma = \mathcal{M}(\mathcal{A})\). Using continuity of measure, one has

\[ \mu(A) = \lim_{n \to \infty} \mu(A_n) = \lim_{n \to \infty} \nu(A_n) = \nu(A). \]

■

**Theorem 4.3** Let \(\mathcal{A}\) be an algebra of sets, \(B \subset X\) such that for any \(\varepsilon > 0\) there exists \(A_\varepsilon \in \mathcal{A}\) with \(\mu_*(B \Delta A_\varepsilon) < \varepsilon\). Then \(B \in \mathcal{A}\).

**Proof.** 1. Since any outer measure is semi-additive, it suffices to prove that for any \(E \subset X\) one has

\[ \mu_*(E) \geq \mu_*(E \cap B) + \mu_*(E \cap B^c). \]

2a. Since \(\mathcal{A} \subset \mathcal{A}\), one has

\[ \mu_*(E \cap A_\varepsilon) + \mu_*(E \cap A_\varepsilon^c) \leq \mu_*(E). \] (14)

2b. Since \(A \subset B \cup (A \Delta B)\) and since the outer measure \(\mu_*\) is monotone and semi-additive, there is an estimate \(|\mu_*(A) - \mu_*(B)| \leq \mu_*(A \Delta B)\) for any \(A, B \subset X\). (C.f. the proof of similar fact for measures above).

2c. It follows from the monotonicity of \(\mu_*\) that

\[ |\mu_*(E \cap A_\varepsilon) - \mu_*(E \cap B)| \leq \mu_*((E \cap A_\varepsilon) \Delta (E \cap B)) \leq \mu(A_\varepsilon \cap B) < \varepsilon. \]
Therefore, \( \mu^*(E \cap A_\varepsilon) > \mu^*(E \cap B) - \varepsilon \).

In the same manner, \( \mu^*(E \cap A_\varepsilon^c) > \mu^*(E \cap B^c) - \varepsilon \).

2d. Using (14), one obtains

\[
\mu^*(E) > \mu^*(E \cap B) + \mu^*(E \cap B^c) - 2\varepsilon.
\]
5 The Lebesgue Measure on the real line $\mathbb{R}^1$

5.1 The Lebesgue Measure of Bounded Sets of $\mathbb{R}^1$

Put $\mathcal{A}$ for the algebra of all finite unions of semi-segments (semi-intervals) on $\mathbb{R}^1$, i.e. all sets of the form

$$A = \bigcup_{j=1}^{k} [a_j, b_j].$$

Define a mapping $\mu : \mathcal{A} \rightarrow \mathbb{R}$ by:

$$\mu(A) = \sum_{j=1}^{k} (b_j - a_j).$$

**Theorem 5.1** $\mu$ is a measure.

**Proof.**
1. All properties including the (finite) additivity are obvious. The only thing to be proved is the $\sigma$-additivity.

Let $(A_j) \subset \mathcal{A}$ be such a countable disjoint family that

$$A = \bigcup_{j=1}^{\infty} A_j \in \mathcal{A}.$$

The condition $A \in \mathcal{A}$ means that $\bigcup A_j$ is a finite union of intervals.

2. For any positive integer $n$,

$$\bigcup_{j=1}^{n} A_j \subset A,$$

hence

$$\sum_{j=1}^{n} \mu(A_j) \leq \mu(A),$$

and

$$\sum_{j=1}^{\infty} \mu(A_j) = \lim_{n \to \infty} \sum_{j=1}^{n} \mu(A_j) \leq \mu(A).$$

3. Now, let $A^\varepsilon$ a set obtained from $A$ by the following construction. Take a connected component of $A$. It is a semi-segment of the form $[s, t)$. Shift slightly on the left its right-hand end, to obtain a (closed) segment. Do it with all components of $A$, in such a way that

$$\mu(A) < \mu(A^\varepsilon) + \varepsilon. \quad (15)$$
Apply a similar procedure to each semi-segment shifting their left end point to the left 

\[ A_j = [a_j, b_j], \text{ and obtain (open) intervals, } A_j^\varepsilon \text{ with} \]

\[ \mu(A_j^\varepsilon) < \mu(A_j) + \frac{\varepsilon}{2^j}. \quad (16) \]

4. By the construction, \( A^\varepsilon \) is a compact set and \( (A_j^\varepsilon) \) its open cover. Hence, there exists a positive integer \( n \) such that

\[ \bigcup_{j=1}^{n} A_j^\varepsilon \supset A^\varepsilon. \]

Thus

\[ \mu(A^\varepsilon) \leq \sum_{j=1}^{n} \mu(A_j^\varepsilon). \]

The formulas (15) and (16) imply

\[ \mu(A) < \sum_{j=1}^{n} \mu(A_j^\varepsilon) + \varepsilon \leq \sum_{j=1}^{n} \mu(A_j) + \sum_{j=1}^{n} \frac{\varepsilon}{2^j} + \varepsilon, \]

thus

\[ \mu(A) < \sum_{j=1}^{\infty} \mu(A_j) + 2\varepsilon. \]

Now, one can apply the Carathéodory’s scheme developed above, and obtain the measure space \((\mathfrak{A}, \hat{\mu})\). The result of this extension is called the Lebesgue measure. We shall denote the Lebesgue measure on \( \mathbb{R}^1 \) by \( m \).

**Exercises.**

1. A one point set is measurable, and its Lebesgue measure is equal to 0.
2. The same for a countable subset in \( \mathbb{R}^1 \). In particular, \( m(\mathbb{Q} \cap [0, 1]) = 0 \).
3. Any open or closed set in \( \mathbb{R}^1 \) is Lebesgue measurable.

**Definition 5.1** Borel algebra of sets, \( \mathfrak{B} \) on the real line \( \mathbb{R}^1 \) is a \( \sigma \)-algebra generated by all open sets on \( \mathbb{R}^1 \). Any element of \( \mathfrak{B} \) is called a Borel set.

**Exercise.** Any Borel set is Lebesgue measurable.

**Theorem 5.2** Let \( E \subset \mathbb{R}^1 \) be a Lebesgue measurable set. Then for any \( \varepsilon > 0 \) there exists an open set \( G \supset E \) such that \( m(G \setminus E) < \varepsilon \).
Proof. Since $E$ is measurable, $m^*(E) = m(E)$. According the definition of an outer measure, for any $\varepsilon > 0$ there exists a cover $A = \bigcup (a_k, b_k) \supset E$ such that

$$m(A) < m(E) + \frac{\varepsilon}{2}.$$ 

Now, put

$$G = \bigcup (a_k - \frac{\varepsilon}{2^{k+1}}, b_k).$$

Problem. Let $E \subset \mathbb{R}^1$ be a bounded Lebesgue measurable set. Then for any $\varepsilon > 0$ there exists a compact set $F \subset E$ such that $m(E \setminus F) < \varepsilon$. (Hint: Cover $E$ with a semi-segment and apply the above theorem to the $\sigma$-algebra of measurable subsets in this semi-segment).

Corollary. For any $\varepsilon > 0$ there exist an open set $G$ and a compact set $F$ such that $G \supset E \supset F$ and $m(G \setminus F) < \varepsilon$.

Such measures are called regular.

5.2 The Lebesgue Measure on the Real Line $\mathbb{R}^1$

We now abolish the condition of boundness.

Definition 5.2 A set $A$ on the real numbers line $\mathbb{R}^1$ is Lebesgue measurable if for any positive integer $n$ the bounded set $A \cap [-n, n)$ is a Lebesgue measurable set.

Definition 5.3 The Lebesgue measure on $\mathbb{R}^1$ is

$$m(A) = \lim_{n \to \infty} m(A \cap [-n, n)).$$

Definition 5.4 A measure is called $\sigma$-finite if any measurable set can be represented as a countable union of subsets each has a finite measure.

Thus the Lebesgue measure $m$ is $\sigma$-finite.

Problem. The Lebesgue measure on $\mathbb{R}^1$ is regular.

5.3 The Lebesgue Measure in $\mathbb{R}^d$

Definition 5.5 We call a $d$-dimensional rectangle in $\mathbb{R}^d$ any set of the form

$$\{x : x \in \mathbb{R}^d : a_i \leq x_i < b_i \}.$$
Using rectangles, one can construct the Lebesque measure in $\mathbb{R}^d$ in the same fashion as we did for the $\mathbb{R}^1$ case.
6 Measurable functions

Let $X$ be a set, $\mathcal{A}$ a $\sigma$-algebra on $X$.

**Definition 6.1** A pair $(X, \mathcal{A})$ is called a measurable space.

**Definition 6.2** Let $f$ be a function defined on a measurable space $(X, \mathcal{A})$, with values in the extended real number system. The function $f$ is called measurable if the set

$$\{ x : f(x) > a \}$$

is measurable for every real $a$.

**Example.**

**Theorem 6.1** The following conditions are equivalent

1. $\{ x : f(x) > a \}$ is measurable for every real $a$. 
2. $\{ x : f(x) \geq a \}$ is measurable for every real $a$. 
3. $\{ x : f(x) < a \}$ is measurable for every real $a$. 
4. $\{ x : f(x) \leq a \}$ is measurable for every real $a$.

**Proof.** The statement follows from the equalities

$$\{ x : f(x) \geq a \} = \bigcap_{n=1}^{\infty} \{ x : f(x) > a - \frac{1}{n} \},$$
$$\{ x : f(x) < a \} = X \setminus \{ x : f(x) \geq a \},$$
$$\{ x : f(x) \leq a \} = \bigcap_{n=1}^{\infty} \{ x : f(x) < a + \frac{1}{n} \},$$
$$\{ x : f(x) > a \} = X \setminus \{ x : f(x) \leq a \}$$

**Theorem 6.2** Let $(f_n)$ be a sequence of measurable functions. For $x \in X$ set

$$g(x) = \sup_{n} f_n(x) (n \in \mathbb{N})$$
$$h(x) = \limsup_{n \to \infty} f_n(x).$$

Then $g$ and $h$ are measurable.
Proof.

\[ \{ x : g(x) \leq a \} = \bigcap_{n=1}^{\infty} \{ x : f_n(x) \leq a \}. \]

Since the LHS is measurable it follows that the RHS is measurable too. The same proof works for inf.

Now

\[ h(x) = \inf g_m(x), \]

where

\[ g_m(x) = \sup_{n \geq m} f_n(x). \]

**Theorem 6.3** Let \( f \) and \( g \) be measurable real-valued functions defined on \( X \). Let \( F \) be real and continuous function on \( \mathbb{R}^2 \). Put

\[ h(x) = F(f(x), g(x)) \quad (x \in X). \]

Then \( h \) is measurable.

**Proof.** Let \( G_a = \{ (u, v) : F(u, v) > a \} \). Then \( G_a \) is an open subset of \( \mathbb{R}^2 \), and thus

\[ G_a = \bigcup_{n=1}^{\infty} I_n \]

where \( (I_n) \) is a sequence of open intervals

\[ I_n = \{ (u, v) : a_n < u < b_n, c_n < v < d_n \}. \]

The set \( \{ x : a_n < f(x) < b_n \} \) is measurable and so is the set

\[ \{ x : (f(x), g(x)) \in I_n \} = \{ x : a_n < f(x) < b_n \} \cap \{ x : c_n < g(x) < d_n \}. \]

Hence the same is true for

\[ \{ x : h(x) > a \} = \{ x : (f(x), g(x)) \in G_a \} = \bigcup_{n=1}^{\infty} \{ x : (f(x), g(x)) \in I_n \}. \]

**Corollories.** Let \( f \) and \( g \) be measurable. Then the following functions are measurable

\[
\begin{align*}
(i) & \quad f + g \\
(ii) & \quad f \cdot g \\
(iii) & \quad |f| \\
(iv) & \quad \frac{f}{g} \quad (\text{if } g \neq 0) \\
(v) & \quad \max\{f, g\}, \min\{f, g\}
\end{align*}
\]

since \( \max\{f, g\} = 1/2(f + g + |f - g|), \min\{f, g\} = 1/2(f + g - |f - g|) \).
6.1 Step functions (simple functions)

**Definition 6.3** A real valued function $f : X \to \mathbb{R}$ is called simple function if it takes only a finite number of distinct values.

We will use below the following notation

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$$

**Theorem 6.4** A simple function $f = \sum_{j=1}^{n} c_j \chi_{E_j}$ is measurable if and only if all the sets $E_j$ are measurable.

**Exercise.** Prove the theorem.

**Theorem 6.5** Let $f$ be real valued. There exists a sequence $(f_n)$ of simple functions such that $f_n(x) \longrightarrow f(x)$ as $n \to \infty$, for every $x \in X$. If $f$ is measurable, $(f_n)$ may be chosen to be a sequence of measurable functions. If $f \geq 0$, $(f_n)$ may be chosen monotonically increasing.

**Proof.** If $f \geq 0$ set

$$f_n(x) = \sum_{i=1}^{n} 2^{-i} \chi_{E_{ni}} + n \chi_{E_n}$$

where

$$E_{ni} = \{x : \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n}\}, \quad F_n = \{x : f(x) \geq n\}.$$

The sequence $(f_n)$ is monotonically increasing, $f_n$ is a simple function. If $f(x) < \infty$ then $f(x) < n$ for a sufficiently large $n$ and $|f_n(x) - f(x)| < 1/2^n$. Therefore $f_n(x) \longrightarrow f(x)$. If $f(x) = +\infty$ then $f_n(x) = n$ and again $f_n(x) \longrightarrow f(x)$.

In the general case $f = f^+ - f^-$, where

$$f^+(x) := \max\{f(x), 0\}, \quad f^-(x) := -\min\{f(x), 0\}.$$

Note that if $f$ is bounded then $f_n \longrightarrow f$ uniformly.
7 Integration

Definition 7.1 A triple \((X, \mathcal{A}, \mu)\), where \(\mathcal{A}\) is a \(\sigma\)-algebra of subsets of \(X\) and \(\mu\) is a measure on it, is called a measure space.

Let \((X, \mathcal{A}, \mu)\) be a measure space. Let \(f : X \mapsto \mathbb{R}\) be a simple measurable function.

\[
f(x) = \sum_{i=1}^{n} c_i \chi_{E_i}(x) \tag{31}
\]

and

\[
\bigcup_{i=1}^{n} E_i = X, \ E_i \cap E_j = \emptyset \ (i \neq j).
\]

There are different representations of \(f\) by means of (31). Let us choose the representation such that all \(c_i\) are distinct.

Definition 7.2 Define the quantity

\[
I(f) = \sum_{i=1}^{n} c_i \mu(E_i).
\]

First, we derive some properties of \(I(f)\).

Theorem 7.1 Let \(f\) be a simple measurable function. If \(X = \bigcup_{j=1}^{k} F_j\) and \(f\) takes the constant value \(b_j\) on \(F_j\) then

\[
I(f) = \sum_{j=1}^{k} b_j \mu(F_j).
\]

Proof. Clearly, \(E_i = \bigcup_{j: b_j = c_i} F_j\).

\[
\sum_{i} c_i \mu(E_i) = \sum_{i=1}^{n} c_i \mu(\bigcup_{j: b_j = c_i} F_j) = \sum_{i=1}^{n} c_i \sum_{j: b_j = c_i} \mu(F_j) = \sum_{j=1}^{k} b_j \mu(F_j).
\]

This show that the quantity \(I(f)\) is well defined.
Theorem 7.2 If $f$ and $g$ are measurable simple functions then

$$I(\alpha f + \beta g) = \alpha I(f) + \beta I(g).$$

Proof. Let $f(x) = \sum_{j=1}^{n} b_j \chi_{F_j}(x)$, $X = \bigcup_{j=1}^{n} F_j$, $g(x) = \sum_{k=1}^{m} c_k \chi_{G_k}(x)$, $X = \bigcup_{k=1}^{n} G_k$.

Then

$$\alpha f + \beta g = \sum_{j=1}^{n} \sum_{k=1}^{m} (\alpha b_j + \beta c_k) \chi_{E_{jk}}(x)$$

where $E_{jk} = F_j \cap G_k$.

Exercise. Complete the proof.

Theorem 7.3 Let $f$ and $g$ be simple measurable functions. Suppose that $f \leq g$ everywhere except for a set of measure zero. Then

$$I(f) \leq I(g).$$

Proof. If $f \leq g$ everywhere then in the notation of the previous proof $b_j \leq c_k$ on $E_{jk}$ and $I(f) \leq I(g)$ follows.

Otherwise we can assume that $f \leq g + \phi$ where $\phi$ is non-negative measurable simple function which is zero every except for a set $N$ of measure zero. Then $I(\phi) = 0$ and

$$I(f) \leq I(g + \phi) = I(f) + I(\phi) = I(g).$$

Definition 7.3 If $f : X \mapsto \mathbb{R}$ is a non-negative measurable function, we define the Lebesgue integral of $f$ by

$$\int f d\mu := \sup I(\phi)$$

where $\sup$ is taken over the set of all simple functions $\phi$ such that $\phi \leq f$.

Theorem 7.4 If $f$ is a simple measurable function then $\int f d\mu = I(f)$.

Proof. Since $f \leq f$ it follows that $\int f d\mu \geq I(f)$.

On the other hand, if $\phi \leq f$ then $I(\phi) \leq I(f)$ and also

$$\sup_{\phi \leq f} I(\phi) \leq I(f)$$

which leads to the inequality

$$\int f d\mu \leq I(f).$$
Definition 7.4 1. If $A$ is a measurable subset of $X \ (A \in \mathfrak{A})$ and $f$ is a non-negative measurable function then we define

$$\int_A f \, d\mu = \int f \chi_A \, d\mu.$$ 

2. 

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu$$

if at least one of the terms in RHS is finite. If both are finite we call $f$ integrable.

Remark. Finiteness of the integrals $\int f^+ \, d\mu$ and $\int f^- \, d\mu$ is equivalent to the finiteness of the integral $\int |f| \, d\mu$.

If it is the case we write $f \in L^1(X, \mu)$ or simply $f \in L^1$ if there is no ambiguity.

The following properties of the Lebesgue integral are simple consequences of the definition. The proofs are left to the reader.

- If $f$ is measurable and bounded on $A$ and $\mu(A) < \infty$ then $f$ is integrable on $A$.
- If $a \leq f(x) \leq b \ (x \in A)$, $\mu(A) < \infty$ then

$$a \mu(A) \leq \int_A f \, d\mu \leq b \mu(A).$$

- If $f(x) \leq g(x)$ for all $x \in A$ then

$$\int_A f \, d\mu \leq \int_A g \, d\mu.$$ 

- Prove that if $\mu(A) = 0$ and $f$ is measurable then

$$\int_A f \, d\mu = 0.$$ 

The next theorem expresses an important property of the Lebesgue integral. As a consequence we obtain convergence theorems which give the main advantage of the Lebesgue approach to integration in comparison with Riemann integration.
Theorem 7.5 Let $f$ be measurable on $X$. For $A \in \mathcal{A}$ define

$$\phi(A) = \int_A f \, d\mu.$$ 

Then $\phi$ is countably additive on $\mathcal{A}$.

**Proof.** It is enough to consider the case $f \geq 0$. The general case follows from the decomposition $f = f^+ - f^-$. If $f = \chi_E$ for some $E \in \mathcal{A}$ then

$$\mu(A \cap E) = \int_A \chi_E \, d\mu,$$

and $\sigma$-additivity of $\phi$ is the same as this property of $\mu$.

Let $f(x) = \sum_{k=1}^n c_k \chi_{E_k}(x), \ \bigcup_{k=1}^n E_k = X$. Then for $A = \bigcup_{i=1}^\infty A_i, \ A_i \in \mathcal{A}$ we have

$$\phi(A) = \int_A f \, d\mu = \int f \chi_A \, d\mu = \sum_{k=1}^n c_k \mu(E_k \cap A)$$

$$= \sum_{k=1}^n c_k \mu(E_k \cap \bigcup_{i=1}^\infty A_i) = \sum_{k=1}^n c_k \mu\left(\bigcup_{i=1}^\infty (E_k \cap A_i)\right)$$

$$= \sum_{k=1}^n c_k \sum_{i=1}^\infty \mu(E_k \cap A_i) = \sum_{i=1}^\infty \sum_{k=1}^n c_k \mu(E_k \cap A_i)$$

(the series of positive numbers)

$$= \sum_{i=1}^\infty \int_{A_i} f \, d\mu = \sum_{i=1}^\infty \phi(A_i).$$

Now consider general positive $f$’s. Let $\varphi$ be a simple measurable function and $\varphi \leq f$. Then

$$\int_A \varphi \, d\mu = \sum_{i=1}^\infty \int_{A_i} \varphi \, d\mu \leq \sum_{i=1}^\infty \phi(A_i).$$

Therefore the same inequality holds for sup, hence

$$\phi(A) \leq \sum_{i=1}^\infty \phi(A_i).$$

Now if for some $i \ \phi(A_i) = +\infty$ then $\phi(A) = +\infty$ since $\phi(A) \geq \phi(A_n)$. So assume that $\phi(A_i) < \infty \forall i$. Given $\varepsilon > 0$ choose a measurable simple function $\varphi$ such that $\varphi \leq f$ and

$$\int_{A_1} \varphi \, d\mu \geq \int_{A_1} f \, d\mu - \varepsilon, \ \int_{A_2} \varphi \, d\mu \geq \int_{A_2} f - \varepsilon.$$
Hence
\[ \phi(A_1 \cup A_2) \geq \int_{A_1 \cup A_2} \varphi d\mu = \int_{A_1} + \int_{A_2} \varphi d\mu \geq \phi(A_1) + \phi(A_2) - 2\varepsilon, \]
so that \( \phi(A_1 \cup A_2) \geq \phi(A_1) + \phi(A_2) \).

By induction
\[ \phi\left( \bigcup_{i=1}^{n} A_i \right) \geq \sum_{i=1}^{n} \phi(A_i). \]

Since \( A \supset \bigcup_{i=1}^{n} A_i \) we have that
\[ \phi(A) \geq \sum_{i=1}^{n} \phi(A_i). \]

Passing to the limit \( n \to \infty \) in the RHS we obtain
\[ \phi(A) \geq \sum_{i=1}^{\infty} \phi(A_i). \]

This completes the proof. \( \blacksquare \)

**Corollary.** If \( A \in \mathcal{A}, \ B \subset A \) and \( \mu(A \setminus B) = 0 \) then
\[ \int_A f d\mu = \int_B f d\mu. \]

**Proof.**
\[ \int_A f d\mu = \int_B f d\mu + \int_{A \setminus B} f d\mu = \int_B f d\mu + 0. \]

\( \blacksquare \)

**Definition 7.5** \( f \) and \( g \) are called equivalent \( (f \sim g \text{ in writing}) \) if \( \mu\{x : f(x) \neq g(x)\} = 0 \).

It is not hard to see that \( f \sim g \) is relation of equivalence.

(i) \( f \sim f \), (ii) \( f \sim g, \ g \sim h \Rightarrow f \sim h \), (iii) \( f \sim g \Leftrightarrow g \sim f \).

**Theorem 7.6** If \( f \in L^1 \) then \( |f| \in L^1 \) and
\[ \left| \int_A f d\mu \right| \leq \int_A |f| d\mu \]

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Proof.

\[ -|f| \leq f \leq |f| \]

**Theorem 7.7 (Monotone Convergence Theorem)**

Let \((f_n)\) be nondecreasing sequence of nonnegative measurable functions with limit \(f\). Then

\[ \int_A f d\mu = \lim_{n \to \infty} \int_A f_n d\mu, \quad A \in \mathcal{A} \]

*Proof.* First, note that \(f_n(x) \leq f(x)\) so that

\[ \lim_{n} \int_A f_n d\mu \leq \int f d\mu \]

It is remained to prove the opposite inequality.

For this it is enough to show that for any simple \(\varphi\) such that \(0 \leq \varphi \leq f\) the following inequality holds

\[ \int_A \varphi d\mu \leq \lim_{n} \int_A f_n d\mu \]

Take \(0 < c < 1\). Define

\[ A_n = \{x \in A : f_n(x) \geq c\varphi(x)\} \]

then \(A_n \subset A_{n+1}\) and \(A = \bigcup_{n=1}^{\infty} A_n\).

Now observe

\[ c \int_A \varphi d\mu = \int_A c\varphi d\mu = \lim_{n \to \infty} \int_{A_n} c\varphi d\mu \leq \]

(this is a consequence of \(\sigma\)-additivity of \(\phi\) proved above)

\[ \leq \lim_{n \to \infty} \int_{A_n} f_n d\mu \leq \lim_{n \to \infty} \int_A f_n d\mu \]

Pass to the limit \(c \to 1\). \(\blacksquare\)

**Theorem 7.8** Let \(f = f_1 + f_2, f_1, f_2 \in L^1(\mu)\). Then \(f \in L^1(\mu)\) and

\[ \int f d\mu = \int f_1 d\mu + \int f_2 d\mu \]
**Proof.** First, let \( f_1, f_2 \geq 0 \). If they are simple then the result is trivial. Otherwise, choose monotonically increasing sequences \((\varphi_{n,1}), (\varphi_{n,2})\) such that \( \varphi_{n,1} \to f_1 \) and \( \varphi_{n,2} \to f_2 \).

Then for \( \varphi_n = \varphi_{n,1} + \varphi_{n,2} \)

\[
\int \varphi_n d\mu = \int \varphi_{n,1} d\mu + \int \varphi_{n,2} d\mu
\]

and the result follows from the previous theorem.

If \( f_1 \geq 0 \) and \( f_2 \leq 0 \) put

\[
A = \{ x : f(x) \geq 0 \}, \quad B = \{ x : f(x) < 0 \}
\]

Then \( f, f_1 \) and \( -f_2 \) are non-negative on \( A \).

Hence

\[
\int_A f_1 = \int_A f d\mu + \int_A (-f_2) d\mu
\]

Similarly

\[
\int_B (-f_2) d\mu = \int_B f_1 d\mu + \int_B (-f) d\mu
\]

The result follows from the additivity of integral. ■

**Theorem 7.9** Let \( A \in \mathcal{A} \), \( (f_n) \) be a sequence of non-negative measurable functions and

\[
f(x) = \sum_{n=1}^{\infty} f_n(x), \quad x \in A
\]

Then

\[
\int_A f d\mu = \sum_{n=1}^{\infty} \int_A f_n d\mu
\]

**Exercise.** Prove the theorem.

**Theorem 7.10 (Fatou’s lemma)**

If \( (f_n) \) is a sequence of non-negative measurable functions defined a.e. and

\[
f(x) = \lim_{n \to \infty} f_n(x)
\]

then

\[
\int_A f d\mu \leq \lim_{n \to \infty} \int_A f_n d\mu
\]

\[ A \in \mathcal{A} \]
Proof. Put \( g_n(x) = \inf_{i \geq n} f_i(x) \)
Then by definition of the lower limit \( \lim_{n \to \infty} g_n(x) = f(x) \).
Moreover, \( g_n \leq g_{n+1}, g_n \leq f_n \). By the monotone convergence theorem
\[
\int_A f d\mu = \lim_n \int_A g_n d\mu = \lim_n \int_A g_n d\mu \leq \lim_n \int_A f_n d\mu.
\]

**Theorem 7.11 (Lebesgue’s dominated convergence theorem)**
Let \( A \in \mathcal{A}, \ (f_n) \) be a sequence of measurable functions such that \( f_n(x) \to f(x) \) \( (x \in A) \).
Suppose there exists a function \( g \in L^1(\mu) \) on \( A \) such that\[ |f_n(x)| \leq g(x) \]
Then
\[
\lim_n \int_A f_n d\mu = \int_A f d\mu.
\]

Proof. From \( |f_n(x)| \leq g(x) \) it follows that \( f_n \in L^1(\mu) \). Since \( f_n + g \geq 0 \) and \( f + g \geq 0 \), by Fatou’s lemma it follows
\[
\int_A (f + g) d\mu \leq \lim_n \int_A (f_n + g)
\]
or
\[
\int_A f d\mu \leq \lim_n \int_A f_n d\mu.
\]
Since \( g - f_n \geq 0 \) we have similarly
\[
\int_A (g - f) d\mu \leq \lim_n \int_A (g - f_n) d\mu
\]
so that
\[
-\int_A f d\mu \leq -\lim_n \int_A f_n d\mu
\]
which is the same as
\[
\int_A f d\mu \geq \lim_n \int_A f_n d\mu
\]
This proves that
\[
\lim_n \int_A f_n d\mu = \lim_n \int_A f_n d\mu = \int_A f d\mu.
\]
8 Comparison of the Riemann and the Lebesgue integral

To distinguish we denote the Riemann integral by \((R) \int_a^b f(x)dx\) and the Lebesgue integral by \((L) \int_a^b f(x)dx\).

**Theorem 8.1** If a function \(f\) is Riemann integrable on \([a, b]\) then it is also Lebesgue integrable on \([a, b]\) and
\[
(L) \int_a^b f(x)dx = (R) \int_a^b f(x)dx.
\]

**Proof.** Boundedness of a function is a necessary condition of being Riemann integrable. On the other hand, every bounded measurable function is Lebesgue integrable. So it is enough to prove that if a function \(f\) is Riemann integrable then it is measurable.

Consider a partition \(\pi_m\) of \([a, b]\) on \(n = 2^m\) equal parts by points \(a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\) and set
\[
\underline{f}_m(x) = \sum_{k=0}^{2^m-1} m_k \chi_k(x), \quad \overline{f}_m(x) = \sum_{k=0}^{2^m-1} M_k \chi_k(x),
\]
where \(\chi_k\) is a characteristic function of \([x_k, x_{k+1})\) clearly,
\[
\underline{f}_1(x) \leq \underline{f}_2(x) \leq \ldots \leq f(x),
\]
\[
\overline{f}_1(x) \geq \overline{f}_2(x) \geq \ldots \geq f(x).
\]

Therefore the limits
\[
f(x) = \lim_{m \to \infty} \underline{f}_m(x), \quad \overline{f}(x) = \lim_{m \to \infty} \overline{f}_m(x)
\]
exist and are measurable. Note that \(\underline{f}(x) \leq f(x) \leq \overline{f}(x)\). Since \(\underline{f}_m\) and \(\overline{f}_m\) are simple measurable functions, we have
\[
(L) \int_a^b \underline{f}_m(x)dx \leq (L) \int_a^b f(x)dx \leq (L) \int_a^b \overline{f}(x)dx \leq (L) \int_a^b \overline{f}_m(x)dx.
\]

Moreover,
\[
(L) \int_a^b \underline{f}_m(x)dx = \sum_{k=0}^{2^m-1} m_k \Delta x_k = s(f, \pi_m)
\]
and similarly
\[(L) \int_a^b \overline{f}_m(x) = \overline{s}(f, \pi_m).\]

So
\[\underline{s}(f, \pi_m) \leq (L) \int_a^b f(x) dx \leq (L) \int_a^b \overline{f}(x) dx \leq \overline{s}(f, \pi_m).\]

Since \(f\) is Riemann integrable,
\[\lim_{m \to \infty} \underline{s}(f, \pi_m) = \lim_{m \to \infty} \overline{s}(f, \pi_m) = (R) \int_a^b f(x) dx.\]

Therefore
\[(L) \int_a^b (\overline{f}(x) - f(x)) dx = 0\]

and since \(\overline{f} \geq f\) we conclude that
\[f = \overline{f} = \underline{f}\] almost everywhere.

From this measurability of \(f\) follows. \(\blacksquare\)
9 $L^p$-spaces

Let $(X, \mathcal{A}, \mu)$ be a measure space. In this section we study $L^p(X, \mathcal{A}, \mu)$-spaces which occur frequently in analysis.

9.1 Auxiliary facts

Lemma 9.1 Let $p$ and $q$ be real numbers such that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ (this numbers are called conjugate). Then for any $a > 0$, $b > 0$ the inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$ 

holds.

Proof. Note that $\varphi(t) := \frac{t^p}{p} + \frac{1}{q} - t$ with $t \geq 0$ has the only minimum at $t = 1$. It follows that

$$t \leq \frac{t^p}{p} + \frac{1}{q}.$$ 

Then letting $t = ab^{-\frac{1}{p-1}}$ we obtain

$$\frac{a^pb^{-q}}{p} + \frac{1}{q} \geq ab^{-\frac{1}{p-1}},$$

and the result follows. ■

Lemma 9.2 Let $p \geq 1$, $a, b \in \mathbb{R}$. Then the inequality

$$|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p).$$

holds.

Proof. For $p = 1$ the statement is obvious. For $p > 1$ the function $y = x^p$, $x \geq 0$ is convex since $y'' \geq 0$. Therefore

$$\left(\frac{|a| + |b|}{2}\right)^p \leq \frac{|a|^p + |b|^p}{2}.$$ ■

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9.2 The spaces $L^p$, $1 \leq p < \infty$. Definition

Recall that two measurable functions are said to be equivalent (with respect to the measure $\mu$) if they are equal $\mu$-almost everywhere.

The space $L^p = L^p(X, \mathcal{A}, \mu)$ consists of all $\mu$-equivalence classes of $\mathcal{A}$-measurable functions $f$ such that $|f|^p$ has finite integral over $X$ with respect to $\mu$.

We set
\[ \|f\|_p := \left( \int_X |f|^p d\mu \right)^{1/p}. \]

9.3 Hölder’s inequality

Theorem 9.3 Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $f$ and $g$ be measurable functions, $|f|^p$ and $|g|^q$ be integrable. Then $fg$ is integrable and the inequality
\[ \int_X |fg| d\mu \leq \left( \int_X |f|^p d\mu \right)^{1/p} \left( \int_X |g|^q d\mu \right)^{1/q}. \]

Proof. It suffices to consider the case
\[ \|f\|_p > 0, \|g\|_q > 0. \]

Let
\[ a = |f(x)||f|^{-1}_p, \quad b = |g(x)||g|^{-1}_q. \]

By Lemma 1
\[ \frac{|f(x)g(x)|}{\|f\|_p \|g\|_q} \leq \frac{|f(x)|^p}{p\|f\|_p^p} + \frac{|g(x)|^q}{q\|g\|_q^q}. \]

After integration we obtain
\[ \|f\|_p^{-1} \|g\|_q^{-1} \int_X |fg| d\mu \leq \frac{1}{p} + \frac{1}{q} = 1. \]

9.4 Minkowski’s inequality

Theorem 9.4 If $f, g \in L^p$, $p \geq 1$, then $f + g \in L^p$ and
\[ \|f + g\|_p \leq \|f\|_p + \|g\|_p. \]
Proof. If \( \|f\|_p \) and \( \|g\|_p \) are finite then by Lemma 2 \( |f + g|^p \) is integrable and \( \|f + g\|_p \) is well-defined.

\[
|f(x) + g(x)|^p = |f(x) + g(x)||f(x) + g(x)|^{p-1} \leq |f(x)||f(x) + g(x)|^{p-1} + |g(x)||f(x) + g(x)|^{p-1}.
\]

Integrating the last inequality and using Hölder’s inequality we obtain

\[
\int_X |f + g|^p d\mu \leq \left( \int_X |f|^p d\mu \right)^{1/p} \left( \int_X |f + g|^{(p-1)q} d\mu \right)^{1/q} + \left( \int_X |g|^p d\mu \right)^{1/p} \left( \int_X |f + g|^{(p-1)q} d\mu \right)^{1/q}.
\]

The result follows. ■

9.5 \( L^p, 1 \leq p < \infty \), is a Banach space

It is readily seen from the properties of an integral and Theorem 9.3 that \( L^p, 1 \leq p < \infty \), is a vector space. We introduced the quantity \( \|f\|_p \). Let us show that it defines a norm on \( L^p, 1 \leq p < \infty \). Indeed,

1. By the definition \( \|f\|_p \geq 0 \).
2. \( \|f\|_p = 0 \implies f(x) = 0 \) for \( \mu \)-almost all \( x \in X \). Since \( L^p \) consists of \( \mu \)-equivalence classes, it follows that \( f \sim 0 \).
3. Obviously, \( \|\alpha f\|_p = |\alpha|\|f\|_p \).
4. From Minkowski’s inequality it follows that \( \|f + g\|_p \leq \|f\|_p + \|g\|_p \).

So \( L^p, 1 \leq p < \infty \), is a normed space.

Theorem 9.5 \( L^p, 1 \leq p < \infty \), is a Banach space.

Proof. It remains to prove the completeness.

Let \( (f_n) \) be a Cauchy sequence in \( L^p \). Then there exists a subsequence \( (f_{n_k}) (k \in \mathbb{N}) \) with \( n_k \) increasing such that

\[
\|f_m - f_{n_k}\|_p < \frac{1}{2^k} \quad \forall m \geq n_k.
\]

Then

\[
\sum_{i=1}^{k} \|f_{n_{i+1}} - f_{n_i}\|_p < 1.
\]

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Let 
\[ g_k := |f_{n_1}| + |f_{n_2} - f_{n_1}| + \ldots + |f_{n_{k+1}} - f_{n_k}|. \]
Then \( g_k \) is monotonically increasing. Using Minkowski’s inequality we have 
\[
\|g_k^p\|_1 = \|g_k\|^p_p \leq \left( \|f_{n_1}\|_p + \sum_{i=1}^{k} \|f_{n_{i+1}} - f_{n_i}\|_p \right)^p < (\|f_{n_1}\|_p + 1)^p.
\]
Put 
\[ g(x) := \lim_k g_k(x). \]
By the monotone convergence theorem 
\[
\lim_k \int_X g_k^p d\mu = \int_A g^p d\mu.
\]
Moreover, the limit is finite since \( \|g_k^p\|_1 \leq C = (\|f_{n_1}\|_p + 1)^p \).

Therefore 
\[
|f_{n_1}| + \sum_{j=1}^{\infty} |f_{n_{j+1}} - f_{n_j}| \text{ converges almost everywhere}
\]
and so does 
\[
f_{n_1} + \sum_{j=1}^{\infty} (f_{n_{j+1}} - f_{n_j}),
\]
which means that 
\[
f_{n_1} + \sum_{j=1}^{N} (f_{n_{j+1}} - f_{n_j}) = f_{n_{N+1}} \text{ converges almost everywhere as } N \to \infty.
\]
Define 
\[ f(x) := \lim_{k \to \infty} f_{n_k}(x) \]
where the limit exists and zero on the complement. So \( f \) is measurable.

Let \( \epsilon > 0 \) be such that for \( n, m > N \) 
\[
\|f_n - f_m\|_p^p = \int_X |f_n - f_m|^p d\mu < \epsilon/2.
\]
Then by Fatou’s lemma 
\[
\int_X |f - f_m|^p d\mu = \int_X \lim_k |f_{n_k} - f_m|^p d\mu \leq \lim_k \int_X |f_{n_k} - f_m|^p d\mu
\]
which is less than \( \epsilon \) for \( m > N \). This proves that 
\[
\|f - f_m\|_p \to 0 \text{ as } m \to \infty. \]

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